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COMPOSITION SERIES RELATIVE TO A MODULE

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Introduction

Let R be an associative ring with identity and let us denote by mod-R the category of all unital right R-modules. For each hereditary torsion theory τ for mod-R and each $M \in$ mod-R Goldman introduced in [5] the concept of a τ-composition series of M as a generalization of composition series. And it was shown in [5] that M has a τ-composition series if and only if M satisfies the a.c.c. and d.c.c. on τ-closed submodules, and all τ-composition series of M, if there exist, have the same length. Any hereditary torsion theory for mod-R is defined (i.e., cogenerated) by some injective right R-module; so if τ is cogenerated by an injective right R-module E, then any τ-composition series of M can be regarded as a composition series relative to a module E.

In this paper for each (not necessarily injective) right R-module U we will introduce the concept of a U-composition series of any right R-module M. And we will generalize those results which have been obtained in [5]. In Section 2 we will show that when U is M-injective, M has a U-composition series if and only if M satisfies the a.c.c. and d.c.c. on U-closed submodules, i.e., $\{L_R \subseteq M_R \mid M/L$ is U-torsionless$\}$, and all U-comosition series of M have the same length (Theorem 2.6 and 2.8, respectively). Moreover, if U is a quasi-injective, M-injective right R-module with endomorphism ring $S = \mathrm{End}(U_R)$, we will show in Section 3 that there exists a kind of mutual relation between U-composition series of M and composition series of ${}_S\mathrm{Hom}(M_R, U_R)$. In particular, it will be proved that M_R has a U-composition series of length n if and only if ${}_S\mathrm{Hom}(M_R, U_R)$ has a composition series of length n (Theorem 3.4). And in Section 4 we will show some necessary and sufficient conditions for ${}_S\mathrm{Hom}(M_R, U_R)$ to be coperfect, noetherian, and of finite length, respectively, in case U is a quasi-injective right R-module with $S = \mathrm{End}(U_R)$ (Theorem 4.1, 4.3 and 4.5, respectively).

1. Preliminaries

For any hereditary torsion theory τ for mod-R and any $M\in$ mod-R a chain of R-submodules

$$M=M_0\supset M_1\supset\cdots\supset M_n=T_\tau(M)$$

where $T_\tau(M)$ denotes the τ-torsion submodule of M, is called a τ-composition series of M if M_{i-1}/M_i is τ-cocritical, i.e., M_{i-1}/M_i is τ-torsionfree and any proper homomorphic image of M_{i-1}/M_i is τ-torsion for $i=1,\dots,n$.

For $M, U\in$ mod-R, M is said to be U-torsion if $\mathrm{Hom}(M_R, U_R)=(0)$, and M is said to be U-torsionless if $M_R\hookrightarrow\prod U_R$ (a direct product of copies of U). Clearly if M is U-torsion and N is U-torsionless, then $\mathrm{Hom}(M_R, N_R)=(0)$. An R-submodule L of M is said to be a U-closed submodule of M if M/L is U-torsionless. The next lemma can be proved without much difficulty.

Lemma 1.1. *For $L, M, U\in$ mod-R with $L\subseteq M$ let us set $M^*=\mathrm{Hom}(M_R, U_R)$. Then, L is a U-closed submodule of M if and only if*

$$L=\mathrm{ann}_M X=\{m\in M\mid f(m)=0 \textit{ for all } f\in X\}$$

for some subset X of M^*, in fact,

$$L=\mathrm{ann}_M\,\mathrm{ann}_{M^*}L$$
$$=\{m\in M\mid f(m)=0 \textit{ for all } f\in M^* \textit{ such that } f(m')=0 \textit{ for all } m'\in L\}.$$

Hence $\bar{L}=\mathrm{ann}_M\,\mathrm{ann}_{M^*}L$ is smallest among all U-closed submodules of M which contain L.

Throughout this paper $\tau_U(M)$ always denotes $\mathrm{ann}_M M^*=\{m\in M\mid f(m)=0$ for all $f\in M^*\}$, where $M^*=\mathrm{Hom}(M_R, U_R)$. According to Lemma 1.1, $\tau_U(M)$ is the smallest U-closed submodule of M. A chain of R-submodules of $M, M_0\supset M_1\supset\cdots\supset M_n$ is said to be a U-chain of length n if M_{i-1}/M_i is not U-torsion for $i=1,\dots,n$. If M has a U-chain of length n, then we denote it by $U\text{-}\dim M_R\geq n$. If there is not any U-chain of length n in M, we denote it by $U\text{-}\dim M_R\ngeq n$. If $U\text{-}\dim M_R\geq n$ and $U\text{-}\dim M_R\ngeq n+1$, then we denote it by $U\text{-}\dim M_R=n$.

Definition. A non-zero right R-module V is said to be U-cocritical if V is U-torsionless and any proper homomorphic image of V is U-torsion. A chain of R-submodules of M

$$M=M_0\supset M_1\supset\cdots\supset M_n=\tau_U(M)$$

is called a U-composition series of M if M_{i-1}/M_i is U-cocritical for $i=1,\dots,n$.

In case U is a cogenerator in mod-R, V is U-cocritical if and only if V is simple.

Hence in such case a U-composition series of M is nothing but a composition series of M.

As usual, M is said to be N-injective if any R-homomorphism of any R-submodule of N into M can be extended to an R-homomorphism of N into M.

Notation. $\Psi(M)=\{N\in \text{mod-}R \mid M \text{ is } N\text{-injective}\}$.

M is said to be quasi-injective if and only if $M\in\Psi(M)$, and M is injective if and only if $\Psi(M)=\text{mod-}R$. The next lemma is very useful.

Lemma 1.2 (Azumaya [1]). *$\Psi(M)$ is closed under taking submodules, homomorphic images and direct sums.*

Throughout this paper any homomorphism will be written on the side opposite the scalars and $\text{End}(M_R)$ denotes the endomorphism ring of M for each $M\in\text{mod-}R$. Thus, if $S=\text{End}(M_R)$, we can regard M as a left S-module for each $M\in\text{mod-}R$. And $X\subset Y$ $(Y\supset X)$ always implies $X\subseteq Y$ and $X\neq Y$ for any two sets X and Y.

2. *U*-composition series

Throughout this section we assume that every module is a right R-module.

Lemma 2.1. *We have the following assertions.*

(1) *Let $(0)\to X\to Y$ be any exact sequence with $Y\in\Psi(U)$. If Y is U-torsion, then so is X.*

(2) *If $X\in\Psi(U)$, then $\tau_U(X)$ is U-torsion.*

(3) *Let $(0)\to X\xrightarrow{\psi} Y\xrightarrow{\phi} Z\to(0)$ be any exact sequence with $Y\in\Psi(U)$. If X and Z both are U-torsionless, then so is Y.*

Proof. (1) Since U is Y-injective, we get the exact sequence $\text{Hom}(Y_R,U_R)\to \text{Hom}(X_R,U_R)\to(0)$. Since $\text{Hom}(Y_R,U_R)=(0)$ by the assumption, we have $\text{Hom}(X_R,U_R)=(0)$, as desired.

(2) If $\tau_U(X)$ is not U-torsion, there is a non-zero R-homomorphism $f:\tau_U(X)\to U$. Since U is X-injective, f can be extended to $h:X\to U$. Then there is an element x in $\tau_U(X)$ such that $h(x)\neq 0$. This contradicts $\tau_U(X)=\text{ann}_X X^*$, where $X^*=\text{Hom}(X_R,U_R)$.

(3) Let y be any non-zero element of Y. If $\phi(y)\neq 0$, there is an R-homomorphism $h:Z\to U$ such that $h\phi(y)\neq 0$. Hence $f=h\phi:Y\to U$ carries y onto a non-zero element of U. Next, assume $\phi(y)=0$. Then $y\in\text{Ker}\,\phi=\text{Im}\,\psi$. Hence there is an element x in X such that $\psi(x)=y$. Since X is U-torsionless, there is an R-homomorphism $g:X\to U$ such that $g(x)\neq 0$. Then, since U is Y-injective, there is an R-homomorphism $f:Y\to U$ such that $g=f\psi$. Therefore $f(y)=f\psi(x)=g(x)\neq 0$. Thus, we conclude that Y is U-torsionless.

Lemma 2.2. *Let $M \in \Psi(U)$. If*

(a) $$M_0 \supset M_1 \supset \cdots \supset M_n$$

is any U-chain of length n in M, then there is a chain of U-closed submodules M_i' of M with length n as follows:

(b) $$M_0' \supset M_1' \supset \cdots \supset M_n'.$$

Proof. Let us put $M_0'/M_0 = \tau_U(M/M_0)$. Then M/M_0' is U-torsionless. Since M_0/M_1 is not U-torsion, so isn't M_0'/M_1 by Lemma 1.2 and (1) of Lemma 2.1. Next, let us put $M_1'/M_1 = \tau_U(M_0'/M_1)$. Then $M_1'/M_1 \subset M_0'/M_1$ since $\mathrm{Hom}((M_0'/M_1)_R, U_R) \neq (0)$, and M_0'/M_1' is U-torsionless. Since $M/M_1' \in \Psi(U)$ by Lemma 1.2, we can easily verify that M_1' is U-closed in M by using (3) of Lemma 2.1. And, since M_1/M_2 is not U-torsion, so isn't M_1'/M_2 by the same reason as above. let us put $M_2'/M_2 = \tau_U(M_1'/M_2)$. Then $M_2'/M_2 \subset M_1'/M_2$ since $\mathrm{Hom}((M_1'/M_2)_R, U_R) \neq (0)$, and M_1'/M_2' is U-torsionless. Therefore, since $M/M_2' \in \Psi(U)$ by Lemma 1.2, and since M_1'/M_2' and M/M_1' each are U-torsionless, M_2' is U-closed in M by (3) of Lemma 2.1. By repeating this argument, if we put $M_i'/M_i = \tau_U(M_{i-1}'/M_i)$ for $i = 1, \ldots, n$, at last we have a chain $M_0' \supset M_1' \supset \cdots \supset M_n'$ such that M_i' is a U-closed submodule of M for each i.

Making use of Lemma 2.2, we can easily verify that when V is U-torsionless and $V \in \Psi(U)$, V is U-cocritical if and only if $U\text{-}\dim V_R = 1$.

Lemma 2.3. *Let M be a U-torsionless right R-module which belongs to $\Psi(U)$ and let N be a non-zero R-submodule of M. Then we have the following assertions.*

(1) *If M is U-cocritical, so is N.*

(2) *If M/N is U-torsion and N is U-cocritical, then M is U-cocritical, too.*

Proof. (1) In this case $U\text{-}\dim M = 1$. Since N is U-torsionless, clearly $U\text{-}\dim N = 1$. On the other hand, since $N \in \Psi(U)$, N is U-cocritical.

(2) We want to show $U\text{-}\dim M_R = 1$. Suppose $U\text{-}\dim M_R \geq 2$. Then there is a chain of length 2, $M_0 \supset M_1 \supset M_2$ such that each M_i is U-closed in M by Lemma 2.2. Let us put $N_i = N \cap M_i$ for $i = 0, 1, 2$. Since $N/N_i = N/(N \cap M_i) \cong (N + M_i)/M_i$ and M/M_i is U-torsionless, N/N_i is also U-torsionless. Since $U\text{-}\dim N_R = 1$ by the assumption, either $N_0 = N_1$ or $N_1 = N_2$ holds. Now, assume $N_0 = N_1$. Then

$$M_0/M_1 \cong (M_0/N_0)/(M_1/N_1) \cong (M_0/(N \cap M_0))/(M_1/(N \cap M_1))$$

$$\cong ((N + M_0)/N)/((N + M_1)/N) \cong (N + M_0)/(N + M_1).$$

And, since $M/(N + M_1) \in \Psi(U)$ by Lemma 1.2 and $M/(N + M_1)$ is U-torsion by the assumption, M_0/M_1 $(\cong (N + M_0)/(N + M_1))$ is also U-torsion by (1) of Lemma 2.1. But, since M_0/M_1 is U-torsionless, we get $M_0 = M_1$, which is a contradiction. Similarly, $N_1 = N_2$ also induces a contradiction. Hence we have $U\text{-}\dim M_R = 1$, and so M is U-cocritical.

For $M \in \text{mod-}R$ let us denote by $\mathscr{L}(M)$ the modular lattice consisting of all R-submodules of M. For each $L \in \mathscr{L}(M)$ let us put $L^c/L = \tau_U(M/L)$. Then L^c is smallest among all U-closed submodules of M which contain L, that is, $L^c = \operatorname{ann}_M \operatorname{ann}_{M^*} L$, where $M^* = \operatorname{Hom}(M_R, U_R)$, according to Lemma 1.1. Hence $L^c = L$ if and only if L is a U-closed submodule of M. And the intersection of an arbitrary family of U-closed submodules of M is again U-closed in M. Indeed, if $\{L_\lambda\}_{\lambda \in \Lambda}$ is a family of U-closed submodules of M, there is an R-monomorphism: $M/\bigcap_{\lambda \in \Lambda} L_\lambda \to \prod_{\lambda \in \Lambda} M/L_\lambda$. But, since $\prod_{\lambda \in \Lambda} M/L_\lambda$ is U-torsionless, so is also $M/\bigcap_{\lambda \in \Lambda} L_\lambda$. That is, $\bigcap_{\lambda \in \Lambda} L_\lambda$ is U-closed in M.

Lemma 2.4. *Let $M \in \Psi(U)$. If L and N are R-submodules of M, then we have*

$$L^c \cap N^c = (L \cap N)^c.$$

Proof. Since $L \cap N \subseteq L$, $(L \cap N)^c \subseteq L^c$. Similarly, $(L \cap N)^c \subseteq N^c$. Hence $(L \cap N)^c \subseteq L^c \cap N^c$.

Next, we want to show first $L_1 \cap L_2^c \subseteq (L_1 \cap L_2)^c$ for any two R-submodules L_1 and L_2 of M. Let $x \in L_1 \cap L_2^c$. Define a map $\psi : (L_1 + L_2)/L_2 \to M/(L_1 \cap L_2)$ by setting $\psi(x_1 + L_2) = x_1 + L_1 \cap L_2$ for all $x_1 \in L_1$. And let $\alpha \in (M/(L_1 \cap L_2))^* = \operatorname{Hom}((M/L_1 \cap L_2))_R, U_R)$. Since $x \in L_1 \cap L_2^c \subseteq L_1$, $x + L_1 \cap L_2 = \psi(x + L_2)$. Then we have

$$\alpha(x + L_1 \cap L_2) = \alpha\psi(x + L_2) = 0.$$

For, suppose $\alpha\psi(x + L_2) \neq 0$. Since U is M/L_2-injective by the assumption and Lemma 1.2, $\alpha\psi$ can be extended to $\beta : M/L_2 \to U$. Hence $\beta(x + L_2) = \alpha\psi(x + L_2) \neq 0$. That is, $x + L_2 \notin \operatorname{ann}_{M/L_2}(M/L_2)^* = L_2^c/L_2$, where $(M/L_2)^* = \operatorname{Hom}((M/L_2)_R, U_R)$, and so $x \notin L_2^c$, which contradicts the choice of x. Therefore $\alpha(x + L_1 \cap L_2) = 0$ for all $\alpha \in (M/(L_1 \cap L_2))^*$. That is to say,

$$x + L_1 \cap L_2 \in \operatorname{ann}_{M/(L_1 \cap L_2)}(M/(L_1 \cap L_2))^* = (L_1 \cap L_2)^c/(L_1 \cap L_2).$$

Thus, we conclude $x \in (L_1 \cap L_2)^c$. Hence we have $L_1 \cap L_2^c \subseteq (L_1 \cap L_2)^c$, as desired.

Now, putting $L_1 = N$ and $L_2 = L$, we get $L^c \cap N \subseteq (L \cap N)^c$ and so $(L^c \cap N)^c \subseteq (L \cap N)^c$. Next, putting $L_1 = L^c$ and $L_2 = N$, we get $L^c \cap N^c \subseteq (L^c \cap N)^c$. Therefore we have that $L^c \cap N^c \subseteq (L \cap N)^c$ and so $L^c \cap N^c = (L \cap N)^c$. Thus, the proof of Lemma 2.4 is completed.

Let us denote by $\mathscr{C}_U(M)$ the set of all U-closed submodules of M, that is, let us set $\mathscr{C}_U(M) = \{L_R \subseteq M_R \mid L^c = L\}$. Since $\mathscr{C}_U(M)$ is closed under taking intersections, we can give a complete lattice structure to $\mathscr{C}_U(M)$ by setting

$$\bigwedge_{\lambda \in \Lambda} \{L_\lambda\} = \bigcap_{\lambda \in \Lambda} L_\lambda \quad \text{and} \quad \bigvee_{\lambda \in \Lambda} \{L_\lambda\} = \Big(\sum_{\lambda \in \Lambda} L_\lambda\Big)^c$$

for every subset $\{L_\lambda\}_{\lambda \in \Lambda}$ of $\mathscr{C}_U(M)$. Moreover, we have the next proposition.

Proposition 2.5. *Let $M \in \Psi(U)$, that is, let U be M-injective. Then $\mathscr{C}_U(M)$ is a complete modular lattice.*

Proof. First, notice that $\mathscr{L}(M)$ is a modular lattice. Let $K, L, N \in \mathscr{C}_U(M)$ with $K \subseteq L$. Then we have that

$$\begin{aligned} L \wedge (K \vee N) &= L^c \cap (K+N)^c \\ &= (L \cap (K+N))^c \qquad \text{by Lemma 2.4} \\ &= (K + (L \cap N))^c = K \vee (L \wedge N). \end{aligned}$$

Hence $\mathscr{C}_U(M)$ is modular, as desired.

Thus, we have seen that $\mathscr{C}_U(M)$ is a complete modular lattice which contains the greatest element M and the smallest element $\tau_U(M)$ in case U is M-injective. In general, if $\mathscr{L}$ is a modular lattice with greatest element 1 and smallest element 0, any maximal chain linking 1 to 0 in $\mathscr{L}$ is called a composition chain of $\mathscr{L}$. Next, let $M \in \Psi(U)$. Then any U-composition series of $M, M = M_0 \supset M_1 \supset \cdots \supset M_n = \tau_U(M)$ is a composition chain of $\mathscr{C}_U(M)$. Indeed, since M/M_i, $M_{i-1}/M_i \in \Psi(U)$ by Lemma 1.2, we can easily show that $M_i \in \mathscr{C}_U(M)$ for each i by using (3) of Lemma 2.1 repeatedly and that this chain is maximal in $\mathscr{C}_U(M)$ by using $U\text{-dim}\, M_{i-1}/M_i = 1$ for each i. Conversely, we can also show that any composition chain of $\mathscr{C}_U(M)$ is a U-composition series of M by using Lemma 2.2 and (3) of Lemma 2.1.

Theorem 2.6. *Let $M \in \Psi(U)$. Then M has a U-composition series if and only if $\mathscr{C}_U(M)$ is noetherian and artinian, that is, M satisfies the a.c.c. and d.c.c. on U-closed submodules.*

Proof. This follows from Proposition 2.5 and [9, Chap. III Proposition 3.5].

Corollary 2.7 (Goldman [5]). *Let τ be any hereditary torsion theory for* mod-R *and let $M \in$* mod-R. *Then M has a τ-composition series if and only if M satisfies the a.c.c. and d.c.c. on τ-closed submodules.*

Theorem 2.8 (A generalization of the Jordan–Hölder Theorem). *Let $M \in \Psi(U)$. Then any two U-composition series of M, if there exist, are equivalent in $\mathscr{C}_U(M)$. That is to say, if*

$$M_R = M_0 \supset M_1 \supset \cdots \supset M_n = \tau_U(M)$$

and

$$M_R = N_0 \supset N_1 \supset \cdots \supset N_r = \tau_U(M)$$

each are U-composition series of M, then we have that $n = r$ and there is a permutation ϱ of $\{1, \ldots, n\}$ such that the intervals $[M_i, M_{i-1}]$ and $[N_{\varrho(i)}, N_{\varrho(i)-1}]$ are projective in $\mathscr{C}_U(M)$ in the sense of [9, *Chap.* III] *for $i = 1, \ldots, n$.*

Proof. This follows from Proposition 2.5 and [9, Chap. III Corollary 3.2].

Remark. If we consider the case where U is an injective cogenerator in mod-R in Theorem 2.8, we get the classical Jordan–Hölder Theorem.

Corollary 2.9 ([5]). *Let τ be any hereditary torsion theory for* mod-R *and let* $M \in$ mod-R. *Then any two τ-composition series of M, if there exist, are equivalent. In particular, all τ-composition series of M have the same length.*

Proof. If τ is cogenerated by an injective right R-module E, any τ-composition series is nothing but an E-composition series.

Let $M \in \Psi(U)$. Then, if M has a U-composition series of length n, we will denote it by U-length $M_R = n$. If M has no U-composition series, we will denote it by U-length $M_R = \infty$. If U-length $M_R = n < \infty$, we will call M a module of finite U-length. Next, let τ be a hereditary torsion theory for mod-R and let $M \in$ mod-R. Then, if M has a τ-composition series of length n, we will denote it by τ-length $M_R = n$ and call M of finite τ-length. Otherwise, it will be denoted by τ-length $M_R = \infty$.

Theorem 2.10. *Let $M \in \Psi(U)$. If M has a U-composition series of length n, then any U-chain of M has finite length t and $t \le n$. In particular, any chain of U-closed submodules of M can be refined to a U-composition series of M.*

Proof. Since U-length $M_R = n$, the length of any composition chain of $\mathscr{C}_U(M)$ is equal to n by Theorem 2.8. Let $L_0 \supset L_1 \supset \cdots \supset L_t$ be any U-chain of M. Then there exists a chain of length t, $L'_0 \supset L'_1 \supset \cdots \supset L'_t$ in $\mathscr{C}_U(M)$ by Lemma 2.2. According to [9, Chap. III Proposition 3.3], this chain can be refined to a composition chain of $\mathscr{C}_U(M)$. Therefore we get $t \le n$.

Theorem 2.11. *Let $M \in \Psi(U)$. M has a U-composition series of length n if and only if there is a maximal U-chain of length n in M. That is to say, U-length $M_R =$ U-dim M_R.*

Proof. *Necessity.* If $M = M_0 \supset M_1 \supset \cdots \supset M_n = \tau_U(M)$ is a U-composition series of length n, this is a U-chain of length n. On the other hand, according to Theorem 2.10 this is a maximal U-chain of M.

Sufficiency. Assume that there is a maximal U-chain of length n in M; say

(a) $\qquad M_0 \supset M_1 \supset \cdots \supset M_n.$

Then by Lemma 2.2 we get a chain of U-closed submodules M'_i of M as follows:

(b) $\qquad M'_0 \supset M'_1 \supset \cdots \supset M'_n.$

Since (a) is maximal, M/M_0 is U-torsion; so M/M'_0 is U-torsion, too. This as well as the fact that M'_0 is U-closed in M, implies $M = M'_0$. Next, let us put $N_0 = M_0 \cap M'_1$.

Then $M_0/N_0 \cong (M_0+M_1')/M_1'$, which is U-torsionless and not equal to (0). For, if $(M_0+M_1')/M_1'=(0)$, $M_0 \subseteq M_1'$. And hence M/M_1' is also U-torsion. So we get $M_1'=M$, which is a contradiction. Next, since U-dim $M_0/M_1=1$ by the maximality of (a), U-dim $M_0/N_0=1$. So U-dim$(M_0+M_1')/M_1'=1$. Therefore $(M_0+M_1')/M_1'$ is U-cocritical. On the other hand, since $M_0'/M_0=\tau_U(M/M_0)$ is U-torsion by (2) of Lemma 2.1, $M_0'/(M_0+M_1')$ is U-torsion, too, as a homomorphic image of M_0'/M_0. Hence M_0'/M_1' is U-cocritical by (2) of Lemma 2.3. Similarly, if we put $N_1=M_1\cap M_2'$, $(M_1+M_2')/M_2'$ $(\cong M_1/N_1)$ is U-cocritical by the same reason as above. And $M_1'/M_1=\tau_U(M_0'/M_1)$ is U-torsion by (2) of Lemma 2.1. And, since $M_1'/(M_1+M_2')$ is a U-torsion module as a homomorphic image of M_1'/M_1, M_1'/M_2' is U-cocritical by (2) of Lemma 2.3. Repeating this argument, we have that M_{i-1}'/M_i' is U-cocritical for $i=1,\dots,n$. Next, since $M_n/(M_n\cap\tau_U(M))$ $(\cong(M_n+\tau_U(M))/\tau_U(M))$ is U-torsionless and (a) is maximal, we have $M_n=M_n\cap\tau_U(M)$; so $M_n\subseteq\tau_U(M)$. Since M_n' is U-closed in M, $\tau_U(M)\subseteq M_n'$. And, since $\tau_U(M_{n-1}'/M_n)=M_n'/M_n$, M_n' is smallest among all U-closed submodules of M_{n-1}' which contain M_n. Hence we have $\tau_U(M)=M_n'$. Therefore M has a U-composition series of length n as follows:

(c) $$M=M_0'\supset M_1'\supset\cdots\supset M_n'=\tau_U(M).$$

This completes the proof of Theorem 2.11.

Theorem 2.12. *Let $(0)\to A\to B\xrightarrow{\phi} C\to(0)$ be any exact sequence of right R-modules with $B\in\Psi(U)$. Then we have*

$$U\text{-length}\, B_R = U\text{-length}\, A_R + U\text{-length}\, C_R.$$

Proof. First, suppose U-length $A=r$ and U-length $C=s$. Let

(a) $$\tau_U(A)=A_0\subset A_1\subset\cdots\subset A_r=A$$

and

(b) $$\tau_U(C)=C_0\subset C_1\subset\cdots\subset C_s=C$$

be U-composition series of A and C, respectively. Let us put $A_{r+j}=\phi^{-1}(C_j)$ for $j=0,1,\dots,s$. Then we get a chain

(c) $$\tau_U(A)=A_0\subset A_1\subset\cdots\subset A_r=A\subseteq A_{r+0}\subset A_{r+1}\subset\cdots\subset A_{r+s}=B.$$

Then A_i/A_{i-1} and A_{r+j}/A_{r+j-1} both are U-cocritical for $i=1,\dots,r$ and $j=1,\dots,s$. Since $A_i/A_{i-1}, A_{r+j}/A_{r+j-1}\in\Psi(U)$ by Lemma 1.2, we have U-dim $A_i/A_{i-1}=1=U$-dim A_{r+j}/A_{r+j-1} for all i and all j. Clearly $\tau_U(C_1)\subseteq\tau_U(C)$. Next, suppose $x\in C_1$ and $x\notin\tau_U(C_1)$. Then there is an R-homomorphism $\alpha: C_1\to U$ such that $\alpha(x)\neq 0$. Since U is C-injective by Lemma 1.2, α can be extended to $\beta: C_R\to U_R$. Hence $x\notin \operatorname{ann}_C C^*=\tau_U(C)$, and so $\tau_U(C)\subseteq\tau_U(C_1)$. Hence $\tau_U(C_1)=\tau_U(C)$. Now, ϕ induces the R-isomorphism $\bar\phi: A_{r+1}/A\to C_1$ with $\bar\phi(A_{r+0}/A)=\tau_U(C)$ and $\bar\phi(\tau_U(A_{r+1}/A))=\tau_U(C_1)$ since $A_{r+1}/A\in\Psi(U)$. Thus, we get $\tau_U(A_{r+1}/A)=A_{r+0}/A$.

Hence $U\text{-dim}\, A_{r+1}/A = U\text{-dim}(A_{r+1}/A)/\tau_U(A_{r+1}/A) = U\text{-dim}(A_{r+1}/A)/(A_{r+0}/A) = U\text{-dim}\, C_1/C_0 = 1$. Thus,

(c′) $\qquad \tau_U(A) = A_0 \subset A_1 \subset \cdots \subset A_r \subset A_{r+1} \subset \cdots \subset A_{r+s} = B$

is a maximal U-chain of length $r+s$. Therefore $U\text{-length}\, B = r+s = U\text{-length}\, A + U\text{-length}\, C$ by Theorem 2.11.

Conversely, suppose $U\text{-length}\, B = n$. According to Theorem 2.10 and 2.11, $U\text{-length}\, A = r$ for some integer $r \leq n$. Let

(d) $\qquad \tau_U(A) = A_0 \subset A_1 \subset \cdots \subset A_r = A$

be a U-composition series of A and let

(e) $\qquad A_0 \subset A_1 \subset \cdots \subset A_r = A_{r+0} \subset A_{r+1} \subset \cdots \subset A_{r+s}$

be a refinement of (d) which is a maximal U-chain of B. Then $n = r+s$ by Theorem 2.11. If we put $\bar{A}_{r+j} = A_{r+j}/A$ for $j = 0, 1, \ldots, s$, then we have a maximal U-chain of $\bar{B} = B/A$ as follows:

(f) $\qquad (0) = \bar{A}_{r+0} \subset \bar{A}_{r+1} \subset \cdots \subset \bar{A}_{r+s}$.

Hence we have $U\text{-length}\, C = U\text{-length}\, \bar{B} = s$ by Theorem 2.11. Therefore $U\text{-length}\, B = n = r+s = U\text{-length}\, A + U\text{-length}\, C$. Thus, the proof of Theorem 2.12 is completed.

Corollary 2.13. *Let $M \in \Psi(U)$ and let M be of finite U-length. Then for any two R-submodules L and N of M we have*

$$U\text{-length}(L+N) + U\text{-length}(L \cap N) = U\text{-length}\, L + U\text{-length}\, N.$$

Proof. Applying Theorem 2.12 to the following two exact sequences

$$(0) \to L \to (L+N) \to (L+N)/L \to (0)$$

and

$$(0) \to L \cap N \to N \to N/(L \cap N) \to (0),$$

we can easily get the required equality.

3. A characterization of modules of finite U-length

In this section we will give a new type of characterization of a module M of finite U-length in case U is a quasi-injective, M-injective right R-module. For $M, U \in$ mod-R with $S = \mathrm{End}(U_R)$ let us set ${}_SM^* = {}_S\mathrm{Hom}(M_R, U_R)$. As usual, we put

$$\mathrm{ann}_M X = \{m \in M \mid f(m) = 0 \text{ for all } f \in X\}$$

for any subset X of M^* and

$$\mathrm{ann}_{M^*} L = \{f \in M^* \mid f(m) = 0 \text{ for all } m \in L\}$$

for any subset L of M. Clearly $\mathrm{ann}_M X$ is an R-submodule of M and $\mathrm{ann}_{M^*} L$ is an S-submodule of M^*.

Lemma 3.1. *Let $V \in \Psi(U)$. If U-length $V_R = 1$, then $V/\tau_U(V)$ is U-cocritical.*

Proof. It is clear by the assumption and the definition of a U-composition series.

Lemma 3.2. *Let $U, V \in \Psi(U)$ with $S = \mathrm{End}(U_R)$ and let ${}_SV^* = {}_S\mathrm{Hom}(V_R, U_R)$. Then, ${}_SV^*$ is simple if and only if U-length $V_R = 1$.*

Proof. *Sufficiency.* The exact sequence $(0) \to \tau_U(V) \to V \to V/\tau_U(V) \to (0)$ induces the exact sequence of left S-modules as follows:

$$(0) \to \mathrm{Hom}((V/\tau_U(V))_R, U_R) \to \mathrm{Hom}(V_R, U_R) \to \mathrm{Hom}(\tau_U(V)_R, U_R) \to (0),$$

because U is V-injective. By (2) of Lemma 2.1, $\tau_U(V)$ is U-torsion. Hence $\mathrm{Hom}(\tau_U(V)_R, U_R) = (0)$. Therefore we have $\mathrm{Hom}((V/\tau_U(V))_R, U_R) \cong \mathrm{Hom}(V_R, U_R)$. Now, since $V \in \Psi(U)$ and U-length $V_R = 1$, $V/\tau_U(V)$ is U-cocritical by Lemma 3.1. Hence we may assume that V is U-cocritical without any loss of generality. Let $0 \neq \alpha \in V^*$, and suppose $\mathrm{Ker}\,\alpha \neq (0)$. Then $\mathrm{Im}\,\alpha \cong V/W$ for some non-zero submodule W of V. Since V is U-cocritical, V/W is U-torsion. On the other hand, $\mathrm{Im}\,\alpha$ is U-torsionless as a submodule of U. Hence $\mathrm{Im}\,\alpha = (0)$, which contradicts $\alpha \neq 0$. Hence we have $\mathrm{Ker}\,\alpha = (0)$. That is, every non-zero R-homomorphism of V into U is a monomorphism. Now, let $\alpha, \beta \in V^*$ with $\alpha \neq 0$. Since U_R is quasi-injective, there is an R-homomorphism $s: U \to U$ such that $\beta = s\alpha$. Hence ${}_SV^* = S\alpha$. That is, ${}_SV^*$ is simple.

Necessity. Assume that ${}_SV^*$ is simple. Then V_R has exactly two U-closed submodules $V = \mathrm{ann}_V(0)$ and $\tau_U(V) = \mathrm{ann}_V V^*$ by Lemma 1.1. Hence U-length $V_R = 1$ by Lemma 2.2. This completes the proof of Lemma 3.2.

Lemma 3.3. *Let U be a quasi-injective right R-module with $S = \mathrm{End}(U_R)$ and let us set ${}_SM^* = {}_S\mathrm{Hom}(M_R, U_R)$ for each $M \in$ mod-R. Then for any finitely generated S-submodule X of M^* we have*

$$X = \mathrm{ann}_{M^*} \mathrm{ann}_M X.$$

Proof. Since ${}_SX$ is finitely generated, we can put $X = \sum_{i=1}^n S\alpha_i$, where $\alpha_i \in M^*$ for $i = 1, \dots, n$. Define a map $\phi: M_R \to \bigoplus^n U_R$ (the direct sum of n copies of U_R) by setting $\phi(m) = (\alpha_1 m, \alpha_2 m, \dots, \alpha_n m)$ for all $m \in M$. Then $\mathrm{Ker}\,\phi = \mathrm{ann}_M X$. Hence ϕ induces the R-monomorphism $h: M/\mathrm{ann}_M X \to \bigoplus^n U$. Now, clearly $X \subseteq \mathrm{ann}_{M^*} \mathrm{ann}_M X$. Next, we want to show $\mathrm{ann}_{M^*} \mathrm{ann}_M X \subseteq X$. Let β be any element of $\mathrm{ann}_{M^*} \mathrm{ann}_M X$. And, define $f: M/\mathrm{ann}_M X \to U$ by setting $f(m + \mathrm{ann}_M X) = \beta(m)$ for all $m \in M$. Then we have the commutative diagram with exact row as follows:

$$\begin{tikzcd}
(0) \arrow[r] & M/\mathrm{ann}_M X \arrow[r, "h"] \arrow[d, "f"'] & \overset{n}{\bigoplus} U \arrow[dl, "g"] \\
& U &
\end{tikzcd}$$

because $\bigoplus^n U \in \Psi(U)$ by Lemma 1.2. Since we can regard

$$\mathrm{Hom}\left(\overset{n}{\bigoplus} U_R, U_R\right) = \overset{n}{\bigoplus} \mathrm{Hom}(U_R, U_R) = \overset{n}{\bigoplus} S,$$

we are able to put $g = (s_1, s_2, \ldots, s_n)$, where $s_i \in S$ for each i. Hence we have that

$$\begin{aligned}
\beta(m) &= f(m + \mathrm{ann}_M X) = gh(m + \mathrm{ann}_M X) \\
&= g\phi(m) = g(\alpha_1 m, \alpha_2 m, \ldots, \alpha_n m) \\
&= \sum_{i=1}^{n} s_i \alpha_i(m) \quad \text{for all } m \in M.
\end{aligned}$$

Hence

$$\beta = \sum_{i=1}^{n} s_i \alpha_i \in \sum_{i=1}^{n} S\alpha_i = X.$$

Thus, we get $\mathrm{ann}_{M^*} \mathrm{ann}_M X \subseteq X$. Therefore we have $X = ann_{M^*} \mathrm{ann}_M X$.

We are now ready to prove our main result.

Theorem 3.4. *Let $U, M \in \Psi(U)$, that is, let U be a quasi-injective, M-injective right R-module with $S = \mathrm{End}(U_R)$. And let us set ${}_SM^* = {}_S\mathrm{Hom}(M_R, U_R)$. If*

(a) $\quad M_R = M_0 \supset M_1 \supset \cdots \supset M_n = \tau_U(M)$

is a U-composition series of length n, then

(a*) $\quad (0) = X_0 \subset X_1 \subset \cdots \subset X_n = {}_SM^*$

where $X_i = \mathrm{ann}_{M^*} M_i$ for each i, is a composition series of length n. Then if we put $M_i' = \mathrm{ann}_M X_i$ for each i,

(a**) $\quad M_R = M_0' \supset M_1' \supset \cdots \supset M_n' = \tau_U(M)$

is equal to (a).

Conversely, if

(b) $\quad (0) = X_0 \subset X_1 \subset \cdots \subset X_n = {}_SM^*$

is a composition series of length n, then

(b*) $\quad M_R = M_0 \supset M_1 \supset \cdots \supset M_n = \tau_U(M)$

where $M_i = \mathrm{ann}_M X_i$ for each i, is a U-composition series of length n. Then if we *put* $X_i' = \mathrm{ann}_{M^*} M_i$ *for each* i,

$$\text{(b**)} \qquad (0) = X_0' \subset X_1' \subset \cdots \subset X_n' = {}_S M^*$$

is equal to (b).

In particular, we have

$$\text{length}\, {}_S M^* = U\text{-length}\, M_R.$$

Proof. First, assume that a chain (a) is a U-composition series of length n. Since U is M-injective, we can easily see that every M_i is U-closed in M by using (3) of Lemma 2.1 repeatedly. Hence $M_i = \mathrm{ann}_M \mathrm{ann}_{M^*} M_i$ for $i = 0, 1, \dots, n$ according to Lemma 1.1. So, if we put $X_i = \mathrm{ann}_{M^*} M_i$ for each i, we get a chain of S-submodules of M^* with length n as follows:

$$\text{(a*)} \qquad (0) = X_0 \subset X_1 \subset \cdots \subset X_n = {}_S M^*.$$

Next, we want to show that (a*) is a composition series of ${}_S M^*$. Let us define a map $\psi: X_i \to \mathrm{Hom}((M_{i-1}/M_i)_R, U_R)$ by setting $[(v)\psi\}(m + M_i) = v(m)$ for all $v \in X_i$ and all $m \in M_{i-1}$. Then clearly ψ is an S-homomorphism and $\mathrm{Ker}\, \psi = X_{i-1}$. Hence ψ induces the S-monomorphism $\bar{\psi}: X_i/X_{i-1} \to \mathrm{Hom}((M_{i-1}/M_i)_R, U_R)$. But, since $M_{i-1}/M_i \in \Psi(U)$ by Lemma 1.2 and M_{i-1}/M_i is U-cocritical, ${}_S\mathrm{Hom}((M_{i-1}/M_i)_R, U_R)$ is simple for each i by Lemma 3.2. Therefore ${}_S(X_i/X_{i-1}) \cong {}_S\mathrm{Hom}((M_{i-1}/M_i)_R, U_R)$; so ${}_S(X_i/X_{i-1})$ is simple for each i. Hence (a*) is a composition series of ${}_S M^*$. And, since $M_i' = \mathrm{ann}_M X_i = \mathrm{ann}_M \mathrm{ann}_{M^*} M_i = M_i$ for each i, (a**) is equal to (a).

Conversely, assume that a chain (b) is a composition series of ${}_S M^*$ with length n. Then, since every X_i is a finitely generated S-submodule of M^*, we have $X_i = \mathrm{ann}_{M^*} \mathrm{ann}_M X_i$ for each i by Lemma 3.3. So, if we put $M_i = \mathrm{ann}_M X_i$ for $i = 0, 1, \dots, n$, we get a chain of length n linking M to $\tau_U(M)$ as follows:

$$\text{(b*)} \qquad M = M_0 \supset M_1 \supset \cdots \supset M_n = \tau_U(M).$$

Next, we want to show that M_{i-1}/M_i is U-cocritical for each i. For this purpose we will first show that ${}_S(\mathrm{ann}_{M_{i-1}^*} M_i)$ $(\cong {}_S\mathrm{Hom}((M_{i-1}/M_i)_R, U_R))$ is simple, where $M_{i-1}^* = \mathrm{Hom}((M_{i-1})_R, U_R)$. Let $\alpha, \beta \in \mathrm{ann}_{M_{i-1}^*} M_i$ with $\alpha \neq 0$. Since U is M-injective, α (resp. β) can be extended to $\bar{\alpha}$ (resp. $\bar{\beta}$) of M^*. Then, since $\bar{\alpha}, \bar{\beta} \in \mathrm{ann}_{M^*} M_i = X_i$ and $\bar{\alpha} \notin \mathrm{ann}_{M^*} M_{i-1} = X_{i-1}$, and since ${}_S(X_i/X_{i-1})$ is simple, there exists an element s of S such that $\bar{\beta} - s\bar{\alpha} \in X_{i-1} = \mathrm{ann}_{M^*} M_{i-1}$. Hence $(\beta - s\alpha)M_{i-1} = (\bar{\beta} - s\bar{\alpha})M_{i-1} = (0)$. Therefore we have $\beta = s\alpha$; so ${}_S(\mathrm{ann}_{M_{i-1}^*} M_i) = S\alpha$. Thus, ${}_S(\mathrm{ann}_{M_{i-1}^*} M_i)$, and hence ${}_S\mathrm{Hom}((M_{i-1}/M_i)_R, U_R)$ is simple, as desired. Hence $U\text{-length}\, M_{i-1}/M_i = 1$ by Lemma 3.2. On the other hand, since M_i is U-closed in M by Lemma 1.1, M_{i-1}/M_i is U-torsionless. Therefore M_{i-1}/M_i is U-cocritical for each i. Thus, (b*) is a U-composition series of M_R. Moreover, since $X_i' = \mathrm{ann}_{M^*} M_i = \mathrm{ann}_{M^*} \mathrm{ann}_M X_i = X_i$ for each i, (b**) is equal to (b). This completes the proof of Theorem 3.4.

Corollary 3.5. *Let* U, S, M *and* M^* *be the same as in Theorem* 3.4. *Suppose*

U-length $M_R = n < \infty$. *Then we have the following statements.*

(1) *For any R-submodule L of M let us put $X = \mathrm{ann}_{M^*} L$. Then*

$$_S(M/L)^* = {}_S\mathrm{Hom}((M/L)_R, U_R) \cong {}_SX$$

and

$$U\text{-length}(M/L)_R = \text{length}\ {}_SX.$$

Moreover,

$$_SL^* = {}_S\mathrm{Hom}(L_R, U_R) \cong {}_S(M^*/X)$$

and

$$U\text{-length}\ L_R = \text{length}\ {}_S(M^*/X) = n - \text{length}\ {}_SX.$$

(2) *For any S-submodule X of M^* let us put $L = \mathrm{ann}_M X$. Then*

$$_SX \cong {}_S(M/L)^* = {}_S\mathrm{Hom}((M/L)_R, U_R)$$

and

$$\text{length}\ {}_SX = U\text{-length}(M/L)_R = n - U\text{-length}\ L_R.$$

Moreover

$$_S(M^*/X) \cong {}_SL^* = {}_S\mathrm{Hom}(L_R, U_R)$$

and

$$\text{length}\ {}_S(M^*/X) = U\text{-length}\ L_R.$$

Proof. (1) It is well known that $_S(M/L)^* = {}_S\mathrm{Hom}((M/L)_R, U_R) \cong {}_S(\mathrm{ann}_{M^*} L) = {}_SX$. Since $M/L \in \Psi(U)$ by Lemma 1.2, we get U-length$(M/L)_R = $ length $_S(M/L)^* =$ length $_SX$ according to Theorem 3.4. Next, the exact sequence

$$(0) \to L \to M \to M/L \to (0)$$

induces the exact sequence

$$(0) \to {}_S(M/L)^* \to {}_SM^* \to {}_SL^* \to (0),$$

because U is M-injective. Hence $_SL^* \cong {}_S(M^*/X)$. Since $L \in \Psi(U)$ by Lemma 1.2, we get U-length $L_R =$ length $_SL^* =$ length $_S(M^*/X) = n -$ length $_SX$ by using Theorem 3.4 again.

(2) According to our assumption and Theorem 3.4 we have length $_SM^* = n$. Hence $_SX$ is finitely generated; so $X = \mathrm{ann}_{M^*} L$ by Lemma 3.3. Therefore $_SX \cong {}_S(M/L)^*$ and $_SL^* \cong {}_S(M^*/X)$ by (1) of this corollary. Hence by Theorem 3.4 and Theorem 2.12 we have length $_SX = U$-length$(M/L)_R = n - U$-length L_R and length $_S(M^*/X) =$ length $_S L^* = U$-length L_R.

Corollary 3.6. *Let τ be a hereditary torsion theory for* mod-R *which is cogenerated by an injective right R-module E with $S = \mathrm{End}(E_R)$. And let us set $_SM^* = {}_S\mathrm{Hom}(M_R, E_R)$ for each $M \in$* mod-R. *Then there is a one-to-one correspondence between τ-composition series of M_R and composition series of $_SM^*$, under which if*

(a) $$M_R = M_0 \supset M_1 \supset \cdots \supset M_n = T_\tau(M)$$

and

(b) $$(0) = X_0 \subset X_1 \subset \cdots \subset X_r = {}_S M^*$$

are the corresponding chains, they satisfy the equality $n = r$ and the conditions $M_i = \mathrm{ann}_M X_i$ and $X_j = \mathrm{ann}_{M^*} M_j$ for all i and all j. Therefore we have

$$\text{length}\ {}_S M^* = \tau\text{-length}\ M_R.$$

Proof. Any τ-composition series of M is nothing but an E-composition series of M and τ-length $M_R = E$-length M_R for each $M \in$ mod-R. Hence this is a direct consequence of Theorem 3.4.

Corollary 3.7. *Let U be a quasi-injective, M-injective cogenerator in* mod-R *with $S = \mathrm{End}(U_R)$. And let us set ${}_S M^* = {}_S\mathrm{Hom}(M_R, U_R)$. Then we have*

$$\text{length}\ {}_S M^* = \text{length}\ M_R.$$

Proof. Since U is a cogenerator in mod-R, every U-composition series is nothing but a composition series. Hence by Theorem 3.4 we have

$$\text{length}\ {}_S M^* = U\text{-length}\ M_R = \text{length}\ M_R.$$

An (a quasi-) injective right R-module U is said to be Δ (resp. Σ) -(quasi-) injective if R satisfies the d.c.c. (resp. a.c.c.) on U-closed right ideals (refer to [3]). Miller–Teply proved in [8] that every Δ-injective module is Σ-injective. Moreover, it was shown in Faith [3] that every Δ-quasi-injective module is Σ-quasi-injective.

Corollary 3.8 (Faith [3, Proposition 8.1]). (1) *Let U be a quasi-injective right R-module with $S = \mathrm{End}(U_R)$. Then the following statements are equivalent.*

(a) *${}_S U$ is of finite length.*

(b) *${}_S U$ is noetherian.*

(c) *U_R is Δ-quasi-injective, that is, $\mathscr{C}_U(R)$ is artinian.*

(2) *In particular, if U_R is an injective module which cogenerates a hereditary torsion theory τ, the following statements are equivalent.*

(a) length ${}_S U = n < \infty$.

(b) *${}_S U$ is noetherian.*

(c) *U_R is Δ-injective, that is, $\mathscr{C}_U(R)$ is artinian.*

(d) τ-length $R_R = n < \infty$.

Proof. (1) Since each $I \in \mathscr{C}_U(R)$ satisfies $I = \mathrm{ann}_R\, \mathrm{ann}_U I$ by Lemma 1.1, (b) implies (c). Next, assume (c). Since every finitely generated S-submodule W of U satisfies $W = \mathrm{ann}_U\, \mathrm{ann}_R W$ by Johnson–Wong's theorem (a special case of Lemma 3.3 for

$M=R$) and since U_R is also Σ-quasi-injective by Miller–Teply–Faith's theorem, (c) implies (a). (a)$\Rightarrow$(b) is trivial.

(2) This follows directly from (1) of this corollary and a special case of Corollary 3.6 for $M=R$.

A ring R is said to be right upper (resp. lower) Levitzki if R satisfies the a.c.c. (resp. d.c.c.) on right annulets. Similarly, a left upper (resp. lower) Levitzki ring is defined. A lower and upper Levitzki ring is called Levitzki for short.

Corollary 3.9. *If there exists a faithful, Δ-quasi-injective module in* mod-R, *then R is a subring of a semi-primary Levitzki ring. If, furthermore, it is balanced, R itself is a semi-primary Levitzki ring.*

Proof. This follows from Corollary 3.8 and [3, Theorem 6.2].

In what follows, we will study endomorphism rings of U-torsionless modules of finite U-length under some additional conditions.

Lemma 3.10. *Let $U, M \in$* mod-R *with M U-torsionless and let $S=\mathrm{End}(U_R)$. Then there is a ring monomorphism:*

$$\mathrm{Hom}(M_R, M_R) \to \mathrm{Hom}({}_S\mathrm{Hom}(M_R, U_R), {}_S\mathrm{Hom}(M_R, U_R)).$$

Proof. Define a map $\psi: \mathrm{Hom}(M_R, M_R) \to \mathrm{Hom}({}_S\mathrm{Hom}(M_R, U_R), {}_S\mathrm{Hom}(M_R, U_R))$ by setting $(f)[\psi(\alpha)] = f\alpha$ for all $\alpha \in \mathrm{Hom}(M_R, M_R)$ and all $f \in \mathrm{Hom}(M_R, U_R)$. Then we can easily verify that ψ is a ring homomorphism. Next, suppose $\psi(\alpha)=0$. Then $f\alpha=0$ for all $f \in \mathrm{Hom}(M_R, U_R)$. Since U cogenerates M, this implies $\alpha=0$. Hence ψ is a ring monomorphism.

Theorem 3.11. *Let $U, M \in \Psi(U)$ with M U-torsionless. Then we have the following assertions.*

(1) *If M is U-cocritical, then the endomorphism ring of M_R is embeddable in a division ring.*

(2) *If U*-length $M_R < \infty$, *then the endomorphism ring of M_R is embeddable in a semi-primary Levitzki ring.*

Proof. Let $S=\mathrm{End}(U_R)$.

(1) Since M is U-cocritical, ${}_S\mathrm{Hom}(M_R, U_R)$ is simple by Lemma 3.2. Hence $\mathrm{Hom}({}_S\mathrm{Hom}(M_R, U_R), {}_S\mathrm{Hom}(M_R, U_R))$ is a division ring. Therefore this result is due to Lemma 3.10.

(2) Since U-length $M_R < \infty$, ${}_S\mathrm{Hom}(M_R, U_R)$ is of finite length by Theorem 3.4. Hence $\mathrm{Hom}({}_S\mathrm{Hom}(M_R, U_R), {}_S\mathrm{Hom}(M_R, U_R))$ is a semiprimary Levitzki ring by [3, Theorem 6.2]. Thus, this follows from Lemma 3.10.

Corollary 3.12. *Let τ be a hereditary torsion theory for* mod-R *and let M be a τ-torsionfree right R-module. Then we have the following assertions.*

(1) *If M is τ-cocritical, then* $\mathrm{End}(M_R)$ *is embeddable in a division ring (see* [4, *Proposition* 18.2]).

(2) *If τ-length $M_R < \infty$, then* $\mathrm{End}(M_R)$ *is embeddable in a semi-primary Levitzki ring.*

4. Modules over the endomorphism ring of a quasi-injective module

Throughout this section let U be a quasi-injective right R-module with $S = \mathrm{End}(U_R)$. For each $M \in$ mod-R let us set ${}_SM^* = {}_S\mathrm{Hom}(M_R, U_R)$. In this section we will show some necessary and sufficient conditions for ${}_SM^*$ to be coperfect, noetherian, and of finite length, respectively. Consequently, we will give some necessary and sufficient conditions for S to be right perfect, left noetherian, and left artinian, respectively. A module M_R is said to be coperfect if M satisfies the d.c.c. on finitely generated R-submodules. It is well known that M_R is coperfect if and only if M_R satisfies the d.c.c. on cyclic submodules (Björk [2]).

Theorem 4.1. ${}_SM^*$ *is coperfect if and only if the a.c.c. holds on*

$$\{L_R \subseteq M_R \mid L = \mathrm{Ker}\,\alpha \text{ for some } \alpha \in M^*\} = \{L_R \subseteq M_R \mid M/L \hookrightarrow U\}.$$

Proof. *Sufficiency.* Let

$$S\alpha \supseteq S(s_1\alpha) \supseteq S(s_2s_1\alpha) \supseteq \cdots$$

be any descending chain of cyclic S-submodules of M^*, where $\alpha \in M^*$ and $s_i \in S$ for each i. Then we have an ascending chain of R-submodules of M as follows:

$$\mathrm{Ker}\,\alpha \subseteq \mathrm{Ker}(s_1\alpha) \subseteq \mathrm{Ker}(s_2s_1\alpha) \subseteq \cdots.$$

By the assumption there exists an integer n such that

$$\mathrm{Ker}(s_ns_{n-1}\cdots s_1\alpha) = \mathrm{Ker}(s_{n+j}s_{n+j-1}\cdots s_1\alpha)$$

for all $j \geq 1$. On the other hand, it holds that

$$\begin{aligned} S(s_is_{i-1}\cdots s_1\alpha) &= \mathrm{ann}_{M^*}\mathrm{ann}_M(s_is_{i-1}\cdots s_1\alpha) \\ &= \mathrm{ann}_{M^*}\mathrm{Ker}(s_is_{i-1}\cdots s_1\alpha) \end{aligned}$$

for all integer $i \geq 1$ by Lemma 3.3. Hence we have

$$\begin{aligned} S(s_ns_{n-1}\cdots s_1\alpha) &= \mathrm{ann}_{M^*}\mathrm{Ker}(s_ns_{n-1}\cdots s_1\alpha) \\ &= \mathrm{ann}_{M^*}\mathrm{Ker}(s_{n+j}s_{n+j-1}\cdots s_1\alpha) \\ &= S(s_{n+j}s_{n+j-1}\cdots s_1\alpha) \end{aligned}$$

for all $j \geq 1$. Hence ${}_SM^*$ satisfies the d.c.c. on cyclic S-submodules. Therefore ${}_SM^*$ is coperfect by Björk's theorem.

Necessity. Consider any ascending chain of kernels of elements α_i of M^* as follows:

$$\operatorname{Ker} \alpha_1 \subseteq \operatorname{Ker} \alpha_2 \subseteq \operatorname{Ker} \alpha_3 \subseteq \cdots.$$

Since U_R is quasi-injective, we have the commutative diagram with exact row as follows:

$$\begin{tikzcd}
(0) \arrow[r] & M/\operatorname{Ker} \alpha_i \arrow[r, "\bar{\alpha}_i"] \arrow[d, "\bar{\alpha}_{i+1}"'] & U \arrow[dl, "s_i"] \\
 & U &
\end{tikzcd}$$

where $\bar{\alpha}_i$ and $\bar{\alpha}_{i+1}$ are the R-maps canonically induced by α_i and α_{i+1}, respectively. Hence we have that

$$\alpha_{i+1}(m) = \bar{\alpha}_{i+1}(m + \operatorname{Ker} \alpha_i) = s_i \bar{\alpha}_i(m + \operatorname{Ker} \alpha_i) = s_i \alpha_i(m)$$

for all $m \in M$. Hence $\alpha_{i+1} = s_i \alpha_i \in S\alpha_i$ for all integer $i \geq 1$. Thus, we get a descending chain of cyclic S-submodules of M^* as follows:

$$S\alpha_1 \supseteq S\alpha_2 \supseteq S\alpha_3 \supseteq \cdots.$$

Since ${}_SM^*$ is coperfect, there exists an integer n such that $S\alpha_n = S\alpha_{n+j}$ for all $j \geq 1$. Then we can easily verify that

$$\operatorname{Ker} \alpha_n = \operatorname{ann}_M(S\alpha_n) = \operatorname{ann}_M(S\alpha_{n+j}) = \operatorname{Ker} \alpha_{n+j}$$

for all $j \geq 1$. This completes the proof of Theorem 4.1.

Corollary 4.2. *S is right perfect if and only if U_R satisfies the a.c.c. on*

$$\{L_R \subseteq U_R \mid L = \operatorname{Ker} s \textit{ for some element } s \in S\} = \{L_R \subseteq U_R \mid U/L \hookrightarrow U\}.$$

The next theorem is an improvement upon a result of Gupta–Varadarajan [6, Proposition 5.3}.

Theorem 4.3. *${}_SM^*$ is noetherian if and only if $\mathscr{C}_U(M)$ is artinian, that is, M_R satisfies the d.c.c on U-closed submodules.*

Proof. First, assume that ${}_SM^*$ is noetherian. Since each $L \in \mathscr{C}_U(M)$ satisfies $L = \operatorname{ann}_M \operatorname{ann}_{M^*} L$ by Lemma 1.1, any strictly descending chain of $\mathscr{C}_U(M)$ induces a strictly ascending chain of S-submodules of M^*. Hence $\mathscr{C}_U(M)$ has to be artinian.

Next, assume that $\mathscr{C}_U(M)$ is artinian. Let $X_1 \subset X_2 \subset X_3 \subset \cdots$ be any strictly ascending chain of finitely generated S-submodules of M^*. According to Lemma 3.3 we have $X_i = \mathrm{ann}_{M^*} \mathrm{ann}_M X_i$ for each i. Hence we get a strictly descending chain of $\mathscr{C}_U(M)$ as follows:

$$\mathrm{ann}_M X_1 \supset \mathrm{ann}_M X_2 \supset \mathrm{ann}_M X_3 \supset \cdots.$$

Hence ${}_SM^*$ satisfies the a.c.c. on finitely generated submodules. Therefore ${}_SM^*$ is noetherian.

Corollary 4.4 (Harada-Ishii [7]). *S is left noetherian if and only if $\mathscr{C}_U(U)$ is artinian, that is, U_R satisfies the d.c.c. on U-closed submodules, i.e., $\{L_R \subsetneq U_R \mid L = \mathrm{ann}_U X$ for some subset X of $S\}$.*

Theorem 4.5. *${}_SM^*$ is of finite length if and only if $\mathscr{C}_U(M)$ is noetherian and artinian, that is, M_R satisfies the a.c.c. and d.c.c. on U-closed submodules.*

Proof. First, assume that ${}_SM^*$ is of finite length. Then by Theorem 4.3 $\mathscr{C}_U(M)$ is artinian. Next, according to Lemma 1.1 any strictly ascending chain of $\mathscr{C}_U(M)$ induces a strictly descending chain of S-submodules of M^*. Since ${}_SM^*$ is artinian, $\mathscr{C}_U(M)$ has to be noetherian. Conversely, assume that $\mathscr{C}_U(M)$ is noetherian and artinian. Then ${}_SM^*$ is noetherian and coperfect by Theorem 4.3 and 4.1, respectively. So ${}_SM^*$ is of finite length.

Remark. For $M, U \in$ mod-R we put $S = \mathrm{End}(U_R)$ and ${}_SM^* = {}_S\mathrm{Hom}(M_R, U_R)$. Let us consider the following conditions.

(a) length ${}_SM^* < \infty$.

(b) $\mathscr{C}_U(M)$ is noetherian and artinian.

(c) M_R has a U-composition series.

If $U \in \Psi(U)$, (a) and (b) are equivalent (Theorem 4.5). If $M \in \Psi(U)$, (b) and (c) are equivalent (Theorem 2.6). And, if $U, M \in \Psi(U)$, all three conditions are equivalent, and in addition we have length ${}_SM^* = U$-length M_R (Theorem 2.6, 4.5 and 3.4).

Corollary 4.6. *S is left artinian if and only if $\mathscr{C}_U(U)$ is noetherian and artinian, that is, U_R satisfies the a.c.c and d.c.c. on U-closed submodules. In fact, we have*

$$\text{length}\ {}_SS = U\text{-length}\ U_R.$$

Corollary 4.7. *Let U be a quasi-injective cogenerator in* mod-R *with* $S = \mathrm{End}(U_R)$. *And let us set ${}_SM^* = {}_S\mathrm{Hom}(M_R, U_R)$ for each $M \in$* mod-R. *Then we have the following assertions.*

(1) *${}_SM^*$ is noetherian if and only if M_R is artinian.*

(2) *${}_SM^*$ is of finite length if and only if so is also M_R.*

(3) *S is left noetherian if and only if U_R is artinian.*

(4) *S is left artinian if and only if U_R is of finite length. Moreover, we have*

$$\text{length}\ {}_SS = \text{length}\ U_R.$$

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